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BLOCKING SUBSPACES BY LINES IN PG(n, q)

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This paper studies the cardinality of a smallest set $\mathcal{T}$ of t-subspaces of the finite projective spaces PG(n,q) such that every s-subspace is incident with at least one element of $\mathcal{T}$, where $0 \le t < s \le n$. This is a very difficult problem and the solution is known only for very few families of triples (s,t,n). When the answer is known, the corresponding blocking configurations usually are partitions of a subspace of PG(n,q) by subspaces of dimension t. One of the exceptions is the solution in the case $t=1$ and $n=2s$. In this paper, we solve the case when $t=1$ and $2s<n\le 3s-3$ and q is sufficiently large.

1. Introduction

This paper continues a research project started in [6]. The frame of the project is the following very general problem of Galois geometry.

Problem. Given a projective space PG(n,q) and integers s,t with $0\le t\le s<n$, what is the smallest cardinality of a set B of t-subspaces with the property that every s-subspace contains an element of B? What is the structure of the smallest sets B?

A survey article on this and related problem was given in [9]. The following cases of this problem have been solved: $t=0$ in [4], $s=n-1$ in [1], $n\le s-1+s/t$ in [3], and finally $t=1$ and $n=2s$ in [6]. For $t=0$, the smallest example consists of the points in a subspace of dimension $n-s$. For $s=n-1$ the solution is the trivial bound, which is $\lceil a/b\rceil$ where a is the number of hyperplanes and b is the number of hyperplanes on a t-subspace. Structural

Mathematics Subject Classification (2000): 51E23, 05B25, 51A05 information on the smallest sets in the case $s=n-1$ has been given in [5]. For $n\leq s-1+s/t$, the smallest example consists of the members of a geometric t-spread in a subspace of dimension $(d+1-s)(t+1)-1$. The example in the case $t=1$ and $n=2s$ is more complicated and will be described below.

It should be noted that in the case $t=0$, in which the above problem is quite easy to solve, much work has been done in order to determine the next smallest minimal examples. For $s=1$, this leads to the theory of the so-called blocking sets. Here especially the plane case has been studied intensively, see [2,7,10].

A line spread of a projective space $\mathrm{PG}(n,q)$ is a set of lines that partition the space. A line spread $\mathcal{S}$ is called *geometric*, if for every subspace T of dimension three that is generated by two lines of spread, the lines of $\mathcal{S}$ contained in T form a spread of T. It is known that a line spread $\mathcal{S}$ of $\mathrm{PG}(2s–1,q)$ is geometric if and only if every subspace of dimension s contains a line of $\mathcal{S}$ (see e.g. [3]). We also consider $\mathrm{PG}(1,q)$, which is the projective line; its only line forms a (geometric) spread of it.

Example 1.1. Consider in $\mathrm{PG}(2s+x-1,q)$, with $s\geq x\geq 1$, subspaces $Y\subseteq A\subseteq C$ with $\dim(Y)=x-1$, $\dim(A)=s+x-1$ and $\dim(C)=s+2x-1$, and let μ be an isomorphism of Y to the quotient geometry C/A. For $P\in Y$, let F_P be the subspace of dimension $s+x$ with $\mu(P)=F_P/A$. Also, let $\mathcal{S}$ be a set of $(q^{2s}-1)/(q^2-1)$ subspaces of dimension $x+1$ on Y that form a geometric line-spread in the quotient geometry on Y. Let B be the set consisting of the following lines. For each $S\in\mathcal{S}$, the q^{2x} lines of S that are skew to Y, and for each $P\in Y$, the $(q^{s+x}-q^{x-1})/(q-1)$ lines of F_P on P that are not contained in Y.

Let U be a subspace of dimension s. We show that U contains a line of B. If $U\cap Y=\emptyset$, then $\langle U,Y\rangle/Y$ is a subspace of dimension s in the quotient geometry on Y, so $\langle U,Y\rangle$ contains an element S of $\mathcal{S}$, and then $U\cap S$ is a line of S that is skew to Y (and thus a line of B). If $c:=\dim(U\cap Y)\geq 0$, then the union of the subspaces F_P with $P\in Y\cap U$ is a subspace T of dimension $s+x+c$; then U meets T in a subspace of dimension at least $c+1$, so for some $P\in U\cap Y$ the subspace F_P contains a point R with $R\in U$ and $R\notin Y$; then PR is a line of B that is contained in U.

As already mentioned, the above problem for $t=1$ has been solved for $n\leq 2s-1$ in [3], and for $n=2s$ in [6]. For $n=2s$, the smallest examples are the ones described in the above example with $x=1$. In this paper we show that the above example is the solution in the case $t=1$ and $2s<n\leq 3s-3$ provided q that is large enough.

Theorem 1.2. *Let B be a set of lines of* PG(n,q) *such that every subspace of dimension s contains a line of B, where $2s<n\le 3s-3$. Put $x:=n-(2s-1)$ and suppose that $q\ge 4^x+2x+1$. Then*

$$|B|\ge \frac{q^{2s}-1}{q^2-1}q^{2x}+\frac{q^x-1}{q-1}\cdot\frac{q^{s+x}-q^{x-1}}{q-1}$$

and the above example is the only one in which equality holds.

Remarks. (a) I am almost convinced that for large q the above example is also best possible for $n=3s-2$ and perhaps for $n=3s-1$. For $s=2$ and $n=3s-2=2s$, this was shown in [6], and for $s=2$ and $n=3s-1=5$, I can also prove it. I am almost convinced that the proof given here can be extended to a proof of the case $n=3s-2$, but the arguments would get quite often more complicated, so I did not try to do this. For the same reason, I did not try to find the best bound for q.

(b) The known solutions of the above problem have usually turned out to be very useful. Especially when also the next best example is known, there are many applications to finite geometry, see the survey article [9].

(c) The construction in Example 1.1 can be generalized to sets B of t-subspaces with the property that every s-subspace contains at least one element of B. We present one example with $t=2$ and $s=4$ in PG$(6,q)$. In PG$(6,q)$, let P be a point, let $\mathcal{S}$ be a set of $(q^6-1)/(q^3-1)$ subspaces of dimension three that all contain P and that form a geometric spread in the quotient geometry on P. Let B_1 be the set consisting of the planes that lie in a member of $\mathcal{S}$ but do not pass through P. Also let B_2 be a set of $(q^6-1)/(q^2-1)$ planes that all contain P and form a geometric line-spread in the quotient geometry on P. Then $|B_1|+|B_2|=(q^6+q^3)+(q^4+q^2+1)$. Every 4-space not containing P contains a plane of B_1, and every 4-space on P contains a plane of B_2. Thus $q^6+q^4+q^3+q^2+1$ planes are enough to block all 4-spaces in PG$(6,q)$. I do not know any smaller example.

2. Point sets in PG(n,q) meeting few subspaces

Throughout this paper, we shall consider the prime power q fixed and use the following notation

$$\Theta_n:=\frac{q^{n+1}-1}{q-1},\quad n\ge -1.$$

This is the number of points in PG(n,q). Given subspaces U and M with $U\subseteq M$, a *complement* of U in M is a subspace U' with $U\cap U'=\emptyset$ and $\langle U,U'\rangle=M$. The following lemma will be used frequently in this paper.

Lemma 2.1. *Let U be a subspace of dimension u of* $\mathrm{PG}(n,q)$.
(a) The number of complements of U in $\mathrm{PG}(n,q)$ *is* $q^{(u+1)(n-u)}$.
(b) Let C be a subspace of U of dimension c. Then the number of subspaces V with $\langle U,V\rangle=\mathrm{PG}(n,q)$ *and* $U\cap V=C$ *is* $q^{(u-c)(n-u)}$.

Proof. (a) is a direct calculation, and (b) follows from (a) applied to the quotient geometry at C. ∎

Lemma 2.2. *Suppose $P_1,\ldots,P_x$ are points of* $\mathrm{PG}(x,q)$, $x\geq 1$, *that span a hyperplane H. Let $A\neq\emptyset$ be a set of points of* $\mathrm{PG}(x,q)\setminus H$, *and let R be the number of lines that meet $\{P_1,\ldots,P_x\}$ and A. Then* $R\geq x|A|^{(x-1)/x}$.

Proof. Let r_i be the number of lines on P_i that meet A. Then $R=r_1+\ldots+r_x$. We use induction on x. For $x=1$, the assertion is trivial. We also prove $x=2$ directly. In this case, $|A|\leq r_1r_2$ and thus $r_1+r_2\geq 2\sqrt{r_1r_2}\geq 2\sqrt{|A|}$.

Now suppose that $x>2$. Let U be the subspace generated by $P_1,\ldots,P_{x-1}$. It has dimension $x-2$. Let $H_1,\ldots,H_q$ be the hyperplanes on U that are different from H. Put $a_i=|A\cap H_i|$. Then $\sum a_i=|A|$. Put $a:=\max\{a_i\}$. By the induction hypothesis, the number of lines that join one of the points $P_1,\ldots,P_{x-1}$ to one of the points of $H_i\cap A$ is at least $(x-1)a_i^{(x-2)/(x-1)}$. Hence

$$r_1+\ldots+r_{x-1}\geq (x-1)\sum_{i=1}^{q}a_i^{\frac{x-2}{x-1}}\geq (x-1)\sum_{i=1}^{q}a_ia^{\frac{-1}{x-1}}=(x-1)|A|a^{\frac{-1}{x-1}}.$$

Since one of the hyperplanes H_i meets A in a points, we have $r_x\geq a$. Thus

$$R=r_1+\ldots+r_x\geq a+(x-1)|A|a^{\frac{-1}{x-1}}.$$

The right hand side, considered as a function in a, obtains its minimum for $a=|A|^{(x-1)/x}$, and this minimum is the lower bound we are looking for. ∎

Remark. If q is a square and if one considers $\mathrm{PG}(n,\sqrt{q})$ embedded in $\mathrm{PG}(n,q)$ and a hyperplane H of $\mathrm{PG}(n,q)$ such that $H\cap\mathrm{PG}(n,\sqrt{q})=\mathrm{PG}(n-1,\sqrt{q})$, then one has equality in the lemma, if $A=\mathrm{PG}(n,\sqrt{q})\setminus H$, and $P_1,\ldots,P_x$ lie in $\mathrm{PG}(n,\sqrt{q})\cap H$.

Lemma 2.3. *Consider in* $\mathrm{PG}(n,q)$ *a subspace X of dimension $x-1\geq 0$ generated by points $P_1,\ldots,P_x$. Let $A\neq\emptyset$ be a set of points in* $\mathrm{PG}(n,q)\setminus X$, *and denote by R the number of lines that meet $\{P_1,\ldots,P_x\}$ and A. Then there is a subspace U of dimension x with $X\subseteq U$ and* $|U\cap A|\geq(x|A|/R)^x$.

Proof. Let a be the maximum number of points of A in the subspaces U of dimension x on X. Consider such a subspace. If R_U is the number of lines that meet $\{P_1,\dots,P_x\}$ and $U\cap A$, then $R_U \ge x|U\cap A|^{(x-1)/x}$ by the previous lemma, provided that $A\cap U\neq\emptyset$. Thus $R_U\ge x|U\cap A|/a^{1/x}$ in any case, that is even if $A\cap U=\emptyset$. Taking the sum over all such subspaces U, we obtain $R\ge x|A|/a^{1/x}$. Hence $a\ge (x|A|/R)^x$. ∎

Lemma 2.4. *Let $x,n\in\mathbb{N}$ with $1\le x\le n$. Suppose that K is a set of $k\le q^x$ points in* PG(n,q) *and suppose that the number of subspaces of codimension x that miss K is at most*

$$q^x\Theta_{x-1}\prod_{j=0}^{x-1}\frac{\Theta_{n-2-j}}{\Theta_j}.$$

Then the average number of secants of K on a point of K is at most $4\Theta_{x-1}$.

Proof. Put

$$S_i=\prod_{j=0}^{x-1}\frac{\Theta_{n-1-i-j}}{\Theta_j},\quad -1\le i\le n-x.$$

Then S_i is the number of subspaces of codimension x of PG(n,q) containing a given subspace of dimension i. The hypothesis states that K meets at least $S_{-1}-q^x\Theta_{x-1}S_1$ subspaces of codimension x. Notice that $S_{i-1}\ge q^xS_i$ for all $0\le i\le n-x$.

Let $\mathcal{C}$ be the set of the subspaces of codimension x that meet K in at least two points. Then K meets $kS_0-\sum_{C\in\mathcal{C}}(|C\cap K|-1)$ subspaces of codimension x. Therefore

$$kS_0-\sum_{C\in\mathcal{C}}(|C\cap K|-1)\ge S_{-1}-q^x\Theta_{x-1}S_1.$$

For $P\in K$, let c_P be the number of subspaces of $\mathcal{C}$ on P, and let r_P be the number of secants of K on P, that is the number of lines on P that meet K in a second point. As each secant of K on P lies in S_1 subspaces of $\mathcal{C}$ while different secants of K on P occur together in S_2 subspaces of $\mathcal{C}$, we have $c_P\ge r_PS_1-\binom{r_P}{2}S_2$. As $r_P<|K|\le q^x$ and $S_1\ge q^xS_2$, then $c_P\ge\frac{1}{2}r_PS_1$. Also

$$\sum_{C\in\mathcal{C}}(|C\cap K|-1)\ge\frac{1}{2}\sum_{C\in\mathcal{C}}|C\cap K|=\frac{1}{2}\sum_{P\in K}c_P\ge\frac{1}{4}\sum_{P\in K}r_PS_1.$$

Hence, if r is the average of the r_P with $P\in K$, then the right hand term is $krS_1/4$. It follows that

$$\frac{1}{4}krS_1\le kS_0-S_{-1}+q^x\Theta_{x-1}S_1.$$

As $S_0 \geq q^x S_1$ and $r < k \leq q^x$, this remains true, if we replace k by q^x. Using then $q^x S_0 - S_{-1} \leq 0$ gives $r \leq 4\Theta_{x-1}$. ∎

Lemma 2.5. *Let $x, n \in \mathbb{N}$ with $1 \leq x \leq n$. Let K be a set of $k \geq \Theta_{x-1}$ points of $\mathrm{PG}(n,q)$ and $w: K \to \mathbb{N}$ a weight function. Put $r := \frac{1}{k}\sum_{P\in K} w(P)$. Then there exist x independent points $P_0, \ldots, P_{x-1} \in K$ such that the sum of the weights of these points is at most xr.*

Proof. Define the points $P_0, \ldots, P_{x-1}$ recursively as follows. Let P_0 be a point with smallest weight. For $i > 0$, let P_i be a point of smallest weight outside the subspace $\langle P_0, \ldots, P_{i-1}\rangle$. Let r_i be the weight of P_i. Then $r_0 \leq r_1 \leq \ldots \leq r_{x-1}$ and

$$kr = \sum_{P\in K} w(P) \geq r_0 + qr_1 + q^2 r_2 + \ldots + q^{x-2} r_{x-2} + (k - \Theta_{x-2}) r_{x-1}.$$

If $r' := (r_0 + \ldots + r_{x-1})/x$, then $\sum_{i=0}^{x-1} r_i q^i \geq \Theta_{x-1} r'$. Since $r_{x-1} \geq r'$ and $k \geq \Theta_{x-1}$, it follows that $kr \geq kr'$. ∎

Lemma 2.6. *Let K be a set of k points in $\mathrm{PG}(n,q)$ where $q^x - \Theta_{x-1} \leq k \leq q^x$ for some $x \in \mathbb{N}$ with $n \geq x \geq 1$. Suppose that at most*

$$q^x \Theta_{x-1} \prod_{j=0}^{x-1} \frac{\Theta_{n-2-j}}{\Theta_j}$$

subspaces of codimension x miss K. If $q \geq 4^x + 2x + 1$, then there exists a subspace of dimension x that meets K in more than Θ_{x-1} points.

Proof. Use the preceding two lemmas to see that there exist points $P_0, \ldots, P_{x-1} \in K$ that span a subspace X of dimension $x-1$ and such that the number of secants of K that meet $\{P_1, \ldots, P_x\}$ is at most $4x\Theta_{x-1}$. Then there exist at most $4x\Theta_{x-1}$ lines that meet $\{P_1, \ldots, P_x\}$ and $A := K \setminus (X \cap K)$. Lemma 2.3 proves the existence of an x-subspace U with $X \subseteq U$ and

$$|U \cap A| \geq \frac{|A|^x}{(4\Theta_{x-1})^x} \ .$$

Hence, if $c := |X \cap K|$, then

$$|U \cap K| \geq c + \frac{(k-c)^x}{(4\Theta_{x-1})^x} \geq \frac{k^x}{(4\Theta_{x-1})^x} \ .$$

Here the last inequality holds, since $k \leq q^x$. We have

$$4^x \Theta_{x-1}^{x+1} (q-1)^{x+1} = 4^x (q^x - 1)^{x+1} \leq 4^x q^{x(x+1)}$$

and

$$\begin{aligned} k^x(q-1)^{x+1} &\geq (q^x-\Theta_{x-1})^x(q-1)^{x+1} = (q^{x+1}-2q^x+1)^x(q-1) \\ &\geq (q^{x+1}-2q^x)^x(q-1) = q^{x^2}(q-2)^x(q-1) \\ &\geq q^{x^2+x-1}(q-2x)(q-1) > q^{x(x+1)}(q-2x-1). \end{aligned}$$

As $q \geq 4^x+2x+1$, it follows that $4^x\Theta_{x-1}^{x+1} < k^x$. This implies that $|U \cap K| > \Theta_{x-1}$. ∎

Lemma 2.7. *Suppose K is a set of Θ_x points of* PG(n,q) *where $1 \leq x < n$. Suppose that there exists a subspace of dimension x that contains more than Θ_{x-1} but not all points of K. Then at least $q^{x(n-x)} - q^{x(n-x-1)}$ subspaces of dimension $n-x$ miss K.*

Proof. By hypothesis, there exists a subspace U of dimension x such that $|K \cap U| = \Theta_x - c$ for some c with $1 \leq c \leq q^x-1$. Then c points of K do not lie in U, and c points of U do not lie in K. Consider a point P of U that is not in K. Then P lies on $q^{x(n-x)}$ subspaces T of dimension $n-x$ with $T \cap U = P$. For each point $R \in K \setminus (U \cap K)$, exactly $q^{x(n-x-1)}$ of these subspaces T contain R. Thus there exist at least $q^{x(n-x)} - cq^{x(n-x-1)}$ subspaces T of dimension $n-x$ such that $T \cap U = P$ and $T \cap K = \emptyset$. As there are c choices for P, we see that K misses at least $c(q^{x(n-x)} - cq^{x(n-x-1)})$ subspaces of dimension $n-x$. It suffices therefore to verify that

$$c(q^{x(n-x)} - cq^{x(n-x-1)}) \geq q^{x(n-x)} - q^{x(n-x-1)}$$

This is equivalent to $(c-1)(q^x-1-c) \geq 0$, which holds as $1 \leq c \leq q^x-1$. ∎

3. Preliminaries

As already noticed, the special case $x=1$ of the main theorem is the main result of [6]. It is stated again in Result 3.1, because it is of great importance for the next section. Result 3.2 is Lemma 2.1 in [6]; it follows quite easily from 3.1.

Result 3.1. *Suppose that B is a set of lines in* PG$(2s,q)$, *$s \geq 1$, such that every s-dimensional subspace contains a line of B. Then*

$$|B| \geq \frac{q^{2s+2}-q^2}{q^2-1} + \frac{q^{s+1}-1}{q-1}.$$

The example described in the introduction (with $x=1$) is the only one in which equality holds.

Result 3.2. *Let C be a set of lines in $\mathrm{PG}(2s-1,q)$, $s\geq 1$, such that every s-space contains a line of C. Let P be a point contained in exactly k lines of C.*

(a) $|C|\geq \frac{q^{2s}-q^2}{q^2-1}+k$.

(b) *If* $k=0$, *then* $|C|\geq \frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}$.

Lemma 3.3. *Let R be a point of an x-space X of $\mathrm{PG}(2s-2+x,q)$ where $s\geq 2$ and $x\geq 0$. Suppose that C is a set of lines with the property that every subspace of dimension $s-1$ that contains no point of $X\setminus\{R\}$ contains a line of C. Let r be the number of lines of C on R.*

(a) $|C|\geq q^{2x}\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)-rq^x+r$.

(b) $|C|\geq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+r$.

Proof. For the proofs, we may assume that the lines of C are disjoint to $X\setminus\{R\}$, so they meet X in R or are skew to X.

(a) By Lemma 2.1 (b), there are $q^{x(2s-2)}$ subspaces of dimension $2s-2$ meeting X in R. By Result 3.1, each of these contains $\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}$ lines of C. Each line of C on R occurs in $q^{x(2s-3)}$ of these subspaces, whereas the lines of C skew to X occur in $q^{x(2s-4)}$ of these subspaces. This gives

$$q^{x(2s-2)}\left(\frac{q^{2s}-q^2}{q^2-1}+\frac{q^s-1}{q-1}\right)\leq (|C|-r)q^{x(2s-4)}+rq^{x(2s-3)}.$$

This proves (a).

(b) Let S be a subspace of dimension $s{+}x$ containing X. Let C' be the set consisting of the lines of C that do not pass through R, and of the $q^x\Theta_{s-1}$ lines of S on R that do not lie in X. Then C' fulfills the hypothesis of the lemma. As R lies on $q^x\Theta_{s-1}$ lines of C', part (a) gives $|C'|\geq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+q^x\Theta_{s-1}$. Hence $|C|\geq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+r$. ∎

Lemma 3.4. *Let R be a point of an x-space X of $\mathrm{PG}(2s-2+x,q)$, $s\geq 2$, $x\geq 0$, and let D be a set of d points outside of X. Suppose that C is a set of lines such that every subspace of dimension $s-1$ that contains no point of $D\cup(X\setminus\{R\})$ contains a line of C. Let r be the number of lines of C on R. Then*

$$|C|\geq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+\max\{0,q^{2x}\Theta_{s-1}-rq^x\}+r-dq^x\Theta_{s-1}.$$

Proof. We may assume that the lines of C are disjoint to $D\cup(X\setminus\{R\})$. For each point P of D choose a subspace U_P of dimension $s+x$ containing P and X, and let C_P be the set consisting of the lines of U_P on P that miss X. Then $|C_P|=\Theta_{s+x-1}-\Theta_x$ and every subspace of dimension $s-1$ on P that is disjoint to X contains a line of C_P. Put

$$D' := \{RP \mid P\in D\} \quad \text{and} \quad C' := C\cup D'\cup\bigcup_{P\in D} C_P.$$

Then every subspace of dimension $s-1$ that contains no point of $X\setminus\{R\}$ contains a line of C'. The point R lies on $r' := r+d'$ lines of C', where $d' := |D'|$. Apply the previous lemma to C' to obtain

$$|C'| \ge q^{2x}\frac{q^{2s}-q^2}{q^2-1} + \max\{0, q^{2x}\Theta_{s-1} - r'q^x\} + r'.$$

As $|C'|\le|C|+d(\Theta_{s+x-1}-\Theta_x)+d'$ and $r'=r+d'$, this gives

$$|C| \ge q^{2x}\frac{q^{2s}-q^2}{q^2-1} + \max\{0, q^{2x}\Theta_{s-1} - r'q^x\} + r - dq^x\Theta_{s-1} + dq^x.$$

As $r'=r+d'\le r+d$, we have

$$\max\{0, q^{2x}\Theta_{s-1} - r'q^x\} + dq^x \ge \max\{0, q^{2x}\Theta_{s-1} - rq^x\}.$$

This proves the lemma. ∎

4. The proof of the main theorem

For the rest of the paper, we consider integers s and x, and a prime power q satisfying

$$s\ge x+2\ge 4 \quad \text{and} \quad q\ge 4^x+2x+1. \tag{1}$$

Also, B denotes a set of lines of PG$(2s-1+x,q)$ with the property that every s-space of PG$(2s-1+x,q)$ contains a line of B. We suppose that

$$|B| \le \frac{q^{2s}-1}{q^2-1}q^{2x}+\Theta_{x-1}q^{x-1}\Theta_s = \frac{q^{2s}-1}{q^2-1}q^{2x}+\Theta_{x-1}q^x\Theta_{s-1}+\Theta_{x-1}q^{x-1} \tag{2}$$

and shall prove in a series of lemmas that equality holds in (2) and that B has the structure described in Example 1.1. This will prove Theorem 1.2.

Lemma 4.1. *Let P be a point. Then P lies on at least $q^x-\Theta_{x-1}$ lines of B. Also the number of $(2s-1)$-subspaces on P that contain no B-line on P is at most $q^x\Theta_{x-1}T_2$ where T_2 is defined in (3).*

Proof. Subspaces of dimension $2s-1$ have codimension x. Thus, the number of $(2s-1)$-spaces containing a given i-subspace is

$$T_i = \prod_{j=0}^{x-1} \frac{\Theta_{2s-2+x-i-j}}{\Theta_j}. \tag{3}$$

This implies that $T_0 > q^x T_1$ and $T_1 > q^x T_2$ and

$$T_0 \geq T_2 \left(q^{2x} + q^x \Theta_{x-1} \left(\frac{1}{\Theta_{2s-3}} + \frac{1}{\Theta_{2s-4}} \right) \right).$$

This last inequality and (2) imply that

$$T_0 \frac{q^{2s}-q^2}{q^2-1} \geq T_2 \left(\frac{q^{2s}-1}{q^2-1} q^{2x} + \Theta_{x-1} q^{x-1} \right) \geq T_2(|B| - \Theta_{x-1} q^x \Theta_{s-1}). \tag{4}$$

Let P be a point, let r be the number of B-lines on P, and let t be the number of $(2s-1)$-spaces on P that contain no B-line on P. By Result 3.2 (b), each of these t spaces contains at least $(q^{2s}-q^2)/(q^2-1)+\Theta_{s-1}$ lines of B. By Result 3.2 (a), the other $(2s-1)$-spaces on P contain at least $(q^{2s}-q^2)/(q^2-1)$ lines of B that do not contain P. Since each of the $|B|-r$ lines of B that does not contain P lies in T_2 subspaces of dimension $2s-1$ on P, it follows that

$$T_0 \frac{q^{2s}-q^2}{q^2-1} + t\Theta_{s-1} \leq (|B|-r)T_2 \leq |B| \cdot T_2.$$

Using (4), it follows that $t \leq q^x \Theta_{x-1} T_2$. Since every B-line on P lies in T_1 subspaces of dimension $2s-1$, we certainly have $t \geq T_0 - rT_1$. If $r \leq q^x$, then $T_0 > q^x T_1$ and $T_1 > q^x T_2$ imply $t \geq q^x(q^x - r)T_2$, which in turn gives $q^x(q^x-r) \leq q^x \Theta_{x-1}$ and thus $r \geq q^x - \Theta_{x-1}$. ∎

Lemma 4.2. *Every point lies on at least q^x lines of B. If the point P lies on exactly q^x lines of B, then there exists a subspace S of dimension $x+1$ on P such that all B-lines on P lie in S.*

Proof. Suppose that P is a point that lies on $q^x - k$ lines of B where $k \geq 0$. Lemma 4.1 gives $k \leq \Theta_{x-1}$. Lemma 2.6, applied to the quotient geometry at P, and Lemma 4.1 ensure the existence of a subspace S of dimension $x+1$ on P that contains more than Θ_{x-1} lines of B on P. Thus, the number of B-lines of S on P is $q^x - k - d$ for some d with $0 \leq d < q^x - k - \Theta_{x-1}$.

Let R_i, $i = 1, \ldots, \Theta_{x-1} + k + d$, be the lines of S on P that are not in B. Also denote by r_i be the number of planes that meet S in R_i and that contain a B-line l with $P \notin l$. Put $r = \min\{r_i\}$.

Let H_i, $i=1,\dots,q^x-k-d$, be the B-lines of S on P. Also denote by h_i the number of B-lines that meet $H_i\setminus\{P\}$ and that are not contained in S. Put $h=\min\{h_i\}$.

Let c_0 be the number of B-lines that are disjoint to S. From our definition we have

$$c_0+(q^x-k-d)h+(\Theta_{x-1}+k+d)r\le|B|.$$

Put $c:=\max\{0,q^x\Theta_{s-1}-r\}$. Then $r\ge q^x\Theta_{s-1}-c$ and hence

$$c_0+(q^x-k-d)h+(\Theta_{x-1}+k+d)(q^x\Theta_{s-1}-c)\le|B|. \tag{5}$$

Consider a line R_i with $r_i=r$ and apply Lemma 3.4 to the quotient geometry at P (where the point R of Lemma 3.4 corresponds to the line R_i). This gives

$$q^{2x}\frac{q^{2s}-q^2}{q^2-1}+\underbrace{\max\{0,q^{2x}\Theta_{s-1}-rq^x\}}_{=cq^x}-dq^x\Theta_{s-1}\le c_0. \tag{6}$$

Combine (2), (5), and (6) to obtain

$$(q^x-k-d)h+c(q^x-d-\Theta_{x-1}-k)+kq^x\Theta_{s-1}\le q^{2x}+\Theta_{x-1}q^{x-1}\ . \tag{7}$$

Using (1) and $q^x-d-k>\Theta_{x-1}$, it follows that $k=0$.

Consider a line H_i with $h_i=h$. It lies on $q^{x(2s-2)}$ subspaces U of dimension $2s-1$ with $U\cap S=H_i$. By Result 3.2, each of these subspaces U contains at least $\frac{q^{2s}-q^2}{q^2-1}$ lines of B that do not contain P. A line that is disjoint to S lies in $q^{x(2s-4)}$ of these subspaces U. A line that meets H_i but is not contained in S lies in $q^{x(2s-3)}$ of these subspaces U. Count pairs (l,U) with lines $l\in B$ and subspaces U of dimension $2s-1$ satisfying $U\cap S=H_i$, and $l\subseteq U$, and $P\notin l$ to obtain

$$q^{x(2s-2)}\frac{q^{2s}-q^2}{q^2-1}\le c_0q^{x(2s-4)}+hq^{x(2s-3)},$$

which gives

$$q^{2x}\frac{q^{2s}-q^2}{q^2-1}\le c_0+hq^x. \tag{8}$$

Combine (2), (5), and (8) using $k=0$ to obtain

$$d(q^x\Theta_{s-1}-c-h)-c\Theta_{x-1}\le q^{2x}+\Theta_{x-1}q^{x-1}=:V. \tag{9}$$

Thus $dq^x\Theta_{s-1}\le V+(h+c)(\Theta_{x-1}+d)$, and (8) gives $(h+c)(q^x-\Theta_{x-1}-d)\le V$. Multiply the first of these inequalities with $q^x-\Theta_{x-1}-d$, and then use the second inequality to obtain $(q^x-\Theta_{x-1}-d)dq^x\Theta_{s-1}\le Vq^x$. As $s\ge x+2$, then $q^x\Theta_{s-1}\ge q^{x+1}\Theta_x\ge qV$. Hence $(q^x-\Theta_{x-1}-d)dq\le q^x$. As $0\le d\le q^x-\Theta_{x-1}-1$ and $q\ge 7$, this implies that $d=0$. ∎

Definition. Points that lie on exactly q^x lines of B will be called *affine* points. The previous lemma shows that for each affine point P there exists an $(x+1)$-space on P that contains all B-lines on P. This $(x+1)$-space on P will be denoted by S_P.

Lemma 4.3. *There exist at least q^{2s+x-1} affine points. There exist at most $2q^{s+2x-1}$ points that are not affine.*

Proof. There are Θ_{2s+x-1} points, each lying on at least q^x lines of B. The number of points that lie on more than q^x lines of B is thus at most $|B|(q+1)-\Theta_{2s+x-1}q^x$. Using the upper bound for $|B|$, we see that this number is at most $2q^{s+2x-1}$. Thus, there are at most this many non-affine points. As $x \le s-2$a and as there are Θ_{2s+x-1} points, it follows that more than q^{2s+x-1} points are affine. ∎

Lemma 4.4. *Let P be an affine point. Let c_0 be the number of B-lines skew to S_P. Consider the B-lines that meet S_P in a unique point; let c_1' be the number of these that meet a B-line on P, and let c_1 be the number of these that do not meet a B-line on P.*

(a)

$$c_1' \le q^{2x-1} - q^{x-1} + \Theta_{x-1}q^{x-2}(q+1),$$
$$c_1 \ge q^x\Theta_{x-1}\Theta_{s-1} - \frac{1}{q-2}\left(\frac{q^{2x}-q^x}{q+1} + \Theta_{x-1}q^{x-1}\right),$$
$$c_0 + c_1' \le \frac{q^{2s}-q^2}{q^2-1}q^{2x} + (q+2)q^{2x-2}.$$

(b) Suppose that the Θ_{x-1} lines of S_P on P that are not in B lie in a subspace of dimension x. Then every line h with $P \in h \subseteq S_P$ and $h \notin B$ lies in at least

$$a := q^x\Theta_{s-1} - \frac{q^{x-1}}{q-2}$$

planes π that contain a B-line and satisfy $\pi \cap S_P = h$. In particular $c_1 \ge a\Theta_{x-1}$.

Proof. Define c_0, c_1', c_1 as in the lemma. Also let c_2 be the number of B-lines that lie in S_P but that do not contain P. Using the upper bound (2) we obtain

$$q^x + c_0 + c_1 + c_1' + c_2 = |B| \le T + q^{2x} + \Theta_{x-1}q^x\Theta_{s-1} + \Theta_{x-1}q^{x-1} \tag{10}$$

where $T := \frac{q^{2s}-q^2}{q^2-1}q^{2x}$. Consider the lines R_i, $i=1,\dots,\Theta_{x-1}$, of S_P on P that are not in B, let r_i be the number of planes π with $\pi \cap S_P = R_i$ that contain a B-line, and put $r = \min r_i$ and $c := \max\{0, q^x\Theta_{s-1} - r\}$. Then

$$c_1 \ge \Theta_{x-1}r \ge \Theta_{x-1}(q^x\Theta_{s-1} - c), \tag{11}$$

and as in (6) the proof of Lemma 4.2 we see that $c_0 \geq T + cq^x$. Therefore

$$c_1' + c_2 + c(q^x - \Theta_{x-1}) \leq q^{2x} - q^x + \Theta_{x-1}q^{x-1}. \tag{12}$$

There are q^{x+1} points Q in S_P for which PQ is a B-line. Each of these lies on at least q^x lines of B, of which at least $q^x - 1$ do not contain P. This implies that $c_2(q+1) + c_1' \geq q^{x+1}(q^x - 1)$, that is

$$c_2 \geq \frac{1}{q+1}(q^{2x+1} - q^{x+1} - c_1'). \tag{13}$$

Using this as a lower bound for c_2 in (12), and using at the same time that $c(q^x - \Theta_{x-1}) > c(q-2)\Theta_{x-1}$, then (12) gives

$$c\Theta_{x-1} \leq \frac{1}{q-2}\left(\frac{q^{2x} - q^x}{q+1} + \Theta_{x-1}q^{x-1} - \frac{qc_1'}{q+1}\right). \tag{14}$$

As $c_1' \geq 0$, this and (11) prove the lower bound for c_1, and since $c \geq 0$, it also proves the upper bound for c_1'. Now combine (10), (11) and (13) to obtain

$$c_0 + \frac{q}{q+1}c_1' \leq T + \frac{q^{2x} - q^x}{q+1} + \Theta_{x-1}q^{x-1} + c\Theta_{x-1}.$$

Combine this with (14) to obtain

$$c_0 + \frac{q}{q+1}c_1'\left(1 + \frac{1}{q-2}\right) \leq T + \frac{q-1}{q-2}\left(\frac{q^{2x} - q^x}{q+1} + \Theta_{x-1}q^{x-1}\right).$$

As the coefficient of c_1' on the left side is larger than one, this implies that $c_0 + c_1' \leq T + (q+2)q^{2x-2}$. This finishes the proof of (a).

In order to prove (b), we suppose now that the Θ_{x-1} lines of S_P on P that are not in B lie in a subspace U of dimension x. Here $P \in U \subseteq S_P$. Thus, every line has at most q points in $S_P \setminus U$ and we therefore can improve (13) to

$$c_2 \geq \frac{1}{q}\sum_{i=0}^{q^{x+1}}(q^x - 1 - f_i) = q^{2x} - q^x - \frac{c_1'}{q}. \tag{15}$$

From (12) and (15) we obtain

$$c(q^x - \Theta_{x-1}) \leq \Theta_{x-1}q^{x-1}.$$

As $q^x - \Theta_{x-1} \geq (q-2)\Theta_{x-1}$, this implies $c \leq q^{x-1}/(q-2)$. As $c = \max\{0, q^x\Theta_{s-1} - r\}$, then $r \geq q^x\Theta_{s-1} - c$. Now the definition of r completes the proof of (b). ∎

Lemma 4.5. *Let P be an affine point.*

(a) If R is an affine point and $R\notin S_P$, then $P\notin S_R$.

(b) If R is an affine point and $R\notin S_P$, then $\dim(S_P\cap S_R)\le x-1$.

(c) There exists a subspace U_P of dimension x with $P\in U_P\subseteq S_P$ such that a line of S_P on P lies in B if and only if it does not lie in U_P.

(d) If R is an affine point then $\dim(U_P\cap U_R)\ge x-1$.

Proof. Let $\mathcal{T}$ be the set consisting of all subspaces T of dimension $2s-1$ for which $T\cap S_P$ is a B-line on P. Apply Lemma 2.1 to see that $|\mathcal{T}|=q^{x(2s-1)}$ and every B-line on P lies in $q^{x(2s-2)}$ elements of $\mathcal{T}$. Moreover, we have the following.

- A u-subspace that meets S_P in a B-line l on P, lies in exactly $q^{x(2s-1-u)}$ elements of $\mathcal{T}$. This follows from Lemma 2.1 applied to the quotient geometry at U.
- A u-subspace U with $U\cap S_P=P$ occurs in $q^{x(2s-1-u)}$ elements of $\mathcal{T}$ (since U and each B-line on P occur together in $q^{x(2s-2-u)}$ elements of $\mathcal{T}$).
- A u-subspace U with $U\cap S_P=\emptyset$ occurs in $q^{x(2s-2-u)}$ elements of $\mathcal{T}$ (since U and each B-line on P occur together in $q^{x(2s-3-u)}$ elements of $\mathcal{T}$).
- If U is a subspace of dimension u and $U\cap S_P$ is a point R with $R\neq P$ and for which PR is a B-line, then U lies in $q^{x(2s-2-u)}$ elements of $\mathcal{T}$.

Let $\mathcal{T}_0$ consist of the subspaces T of $\mathcal{T}$ that have the property that for some point R of T, no B-line on R lies in T. By Result 3.2, each element of $\mathcal{T}$ contains at least $\frac{q^{2s}-q^2}{q^2-1}+1$ lines of B, and each element of $\mathcal{T}_0$ contains at least $\frac{q^{2s}-q^2}{q^2-1}+\Theta_{s-1}$ lines of B. Let $\mathcal{L}$ be the set consisting of the B-lines that either miss S_P, or that meet S_P in a unique point $R\neq P$ for which PR is a B-line. Then each $T\in\mathcal{T}$ contains one B-line on P while the remaining B-lines of T lie in $\mathcal{L}$. We have seen above that each line of $\mathcal{L}$ occurs in $q^{x(2s-3)}$ elements of $\mathcal{T}$. Count incident pairs (l,T) with $l\in\mathcal{L}$ and $T\in\mathcal{T}$ to obtain

$$\left(q^{x(2s-1)}-|\mathcal{T}_0|\right)\frac{q^{2s}-q^2}{q^2-1}+|\mathcal{T}_0|\left(\frac{q^{2s}-q^2}{q^2-1}+\Theta_{s-1}-1\right)\le|\mathcal{L}|q^{x(2s-3)}.$$

In the notation of Lemma 4.4, we have $|\mathcal{L}|=c_0+c_1'$. Use the upper bound for c_0+c_1' given there to obtain

$$|\mathcal{T}_0|(\Theta_{s-1}-1)\le q^{2x-2}(q+2)q^{x(2s-3)}.$$

This implies that

$$|\mathcal{T}_0|\le(q+1)q^{2xs-x-s-1}. \qquad (16)$$

(a) Suppose that R is an affine point with $R \notin S_P$ and assume that $P \in S_R$. As the B-lines on P lie in S_P, then the line PR is not in B. The line PR lies in $q^{x(2s-2)}$ elements T of $\mathcal{T}$. If such a subspace T meets S_R not only in PR, then it contains a plane π of S_R on PR. The line PR lies in Θ_{x-1} planes π of S_R. Such a plane π meets S_P only in P or in a line on P; in both cases, π occurs in at most $q^{x(2s-3)}$ elements of T. Therefore at least $q^{x(2s-2)} - \Theta_{x-1}q^{x(2s-3)}$ elements T of $\mathcal{T}$ satisfy $T \cap S_R = PR$. For such a T, the point R lies on no B-line of T, because the B-lines on R lie in S_R; hence $T \in \mathcal{T}_0$. This shows that

$$|\mathcal{T}_0| \geq q^{x(2s-2)} - \Theta_{x-1}q^{x(2s-3)}$$

As $s \geq x+2$, this contradicts (16).

(b) Let R be an affine point with $R \notin S_P$ and assume that $U := S_P \cap S_R$ has dimension at least x. Then U has dimension x. As $P \notin S_R$ and $R \notin S_P$, then $P, R \notin U$. The B-lines on P and R meet U. Since P and R lie on q^x lines of B, it follows that at least $q^x - \Theta_{x-1}$ points of U are joined to P and R by a line of B. Each of these points lies on at least q^x lines of B, so each of these points lies on at least $q^x - \Theta_{x-1}$ lines of B that do not lie in U. Thus, if we consider all the points of U that are joined to P and R by B-lines, then at least $(q^x - \Theta_{x-1})^2$ lines of B contain one of these points and are not contained in $U = S_P \cap S_R$. We may assume that at least one half of them is not contained in S_P. Then Lemma 4.4 (a) gives

$$\frac{1}{2}(q^x - \Theta_{x-1})^2 \leq q^{2x-1} - q^{x-1} + \Theta_{x-1}q^{x-2}(q+1).$$

This is a contradiction.

(c) For every affine point R, call the Θ_{x-1} lines of S_R on R that are not in B, the *special* lines of R. For affine points R with $R \notin S_P$ denote by f_R the number of special lines h of R that miss all specials lines of P. Consider an affine point R outside S_P and suppose that $f_R > 0$. Let h be a special line of R that meets no special line of P. Then $h \cap S_P = \emptyset$, or h meets some B-line on P in a point distinct from P. In both cases, h lies on $q^{x(2s-3)}$ elements T of $\mathcal{T}$. If such a subspace T meets S_R not only in h, then it contains a plane π of S_R on h. A plane π with $h \subset \pi \subseteq S_R$ does not contain P (since $R \notin S_P$ and hence $P \notin S_R$) and thus lies in at most $q^{x(2s-4)}$ elements of $\mathcal{T}$. As h lies in Θ_{x-1} planes π of S_R, it follows that $\mathcal{T}$ contains at least $q^{x(2s-3)} - \Theta_{x-1}q^{x(2s-4)}$ elements T satisfying $T \cap S_R = h$. These have the property that R lies on no B-line of T, so that $T \in \mathcal{T}_0$. Thus R lies on at least $f_R q^{x(2s-4)}(q^x - \Theta_{x-1})$ elements of $\mathcal{T}_0$.

Count incident pairs (R,T) with affine points R outside S_P and subspaces $T\in\mathcal{T}_0$. As each subspace T of $\mathcal{T}_0$ has $q^2\Theta_{2s-3}$ points outside S_P, it follows that

$$q^{x(2s-4)}(q^x-\Theta_{x-1})\sum_{\substack{R\notin S_P\\ R\text{ affine}}} f_R\le q^{2xs-x-s-1}(q+1)q^2\Theta_{2s-3}\ .$$

Using (1), we obtain $$\sum_{\substack{R\notin S_P\\ R\text{ affine}}} f_R\le q^{s+2x-2}(q+4).$$

By Lemma 4.3 there are at least q^{2s+x-1} affine points. At most Θ_{x+1} lie in S_P and at most $q^{s+2x-2}(q+4)$ points R_i satisfy $f_i>0$. As $2s+x-1\ge s+2x+1$, it follows that there exists an affine point R outside S_P such that each of the Θ_{x-1} special lines on R meets a special line on P. As the intersection points are in S_P and S_R and as $\dim(S_P\cap S_R)\le x-1$ (that was proved in (b)), it follows that $S_P\cap S_R$ has dimension $x-1$ and that all special lines of P lie in $U_P:=\langle S_P\cap S_R,P\rangle$.

(d) We first consider the case when $R\notin S_P$. Assume that $\dim(U_P)\cap\dim(U_R)\le x-2$. Then at least q^{x-1} of the lines of U_R on R miss U_P. Using the notation of the proof of (c), this gives $f_R\ge q^{x-1}$. Again from the proof of (c) we see that R lies on $f_Rq^{x(2s-4)}(q^x-\Theta_{x-1})$ elements of $\mathcal{T}_0$. Hence

$$q^{x-1}(q^{x(2s-3)}-\Theta_{x-1}q^{x(2s-4)})\le|\mathcal{T}_0|\le q^{2xs-x-s-1}(q+1).$$

As $s\ge x+2$, this is a contradiction.

Now we consider the general situation. As $\langle S_P,S_R\rangle$ has dimension at most $2x+3<2s\le 2s-1+x$, Lemma 4.3 proves the existence of an affine point A with $A\notin\langle S_P,S_R\rangle$. Then the special case already proved shows that $U_P\cap U_A$ and $U_R\cap U_A$ have dimension $x-1$. As U_A has dimension x and is not contained in $\langle U_P,U_R\rangle$ (since $A\in U_A$), it follows that $U_P\cap U_A=U_R\cap U_A$. Thus $U_P\cap U_R$ contains $U_P\cap U_A$ and therefore $\dim(U_P\cap U_R)\ge x-1$. ∎

Lemma 4.6. *There exists a subspace Y of dimension x–1 with the following properties.*

(a) If P is an affine point, then $P\notin Y$ and $Y\subseteq S_P$ and furthermore a line of S_P on P is in B iff it misses Y.

(b) If P and R are affine points with $\langle P,Y\rangle\ne\langle R,Y\rangle$, then $S_P=S_R$ or $S_P\cap S_R=Y$.

Proof. (a) It follows from Lemma 4.5 (d) that all subspaces U_P, P an affine point, lie in a common subspace of dimension $x+1$, or contain a common subspace of dimension $x-1$. As $P\in U_P$ for every affine point P and as there are more than Θ_{x+1} affine points (Lemma 4.3), the first case is not possible. Thus there exists a subspace Y of dimension $x-1$ with $Y \subseteq U_P \subseteq S_P$ for all affine points P.

Consider an affine point P. Then there exists an affine point R with $R \notin S_P$. Lemma 4.5 shows that $P \notin S_R$. As $Y \subseteq S_R$, it follows that $P \notin Y$. Thus Y contains no affine point. Then $U_P = \langle Y, P \rangle$, which means that a line of S_P on P lies in B if and only if it misses Y.

(b) Suppose that P and R are affine points with $\langle P, Y \rangle \neq \langle R, Y \rangle$. Then $\langle P, Y \rangle \cap \langle R, Y \rangle = Y$. If $R \notin S_P$, then Lemma 4.5 (b) shows that $S_P \cap S_R = Y$. If $R \in S_P$, then 4.5 (a) gives $P \in S_R$, which implies that $S_P = \langle Y, P, R \rangle = S_R$. ∎

Lemma 4.7. *(a) Every point Q of Y lies on at least $q^x \Theta_{s-1} - \frac{q^{x-1}}{q-2}$ lines of B that meet Y only in Q.*

(b) The number of B-lines that miss Y is at most

$$\frac{q^{2s}-1}{q^2-1} q^{2x} + \frac{q^{2x-1}}{q-2}.$$

Proof. Recall that Y does not contain affine points. Also, if P and P' are affine points with $P' \notin S_P$, then $S_P \cap S_{P'} = Y$ by Lemma 4.6. Thus, it is possible to find affine points $P_1, \ldots, P_t$ for a suitable number t, such that any two subspaces S_{P_i} meet in Y and such that every affine point lies in (a necessarily unique) subspace S_{P_i}. As there are $q^x \Theta_{2s-1}$ points outside Y and as $|S_{P_i} \setminus Y| = q^x(q+1)$, we have $t \leq \Theta_{2s-1}/(q+1)$. Thus $q^{2s-1+x}/t > q^{x+1} - q^{x-1}$. Then Lemma 4.3 shows that one of the sets S_{P_i} contains more than $q^{x+1} - q^{x-1}$ affine points.

Thus, there exists an affine point P such that $S := S_P$ contains more than $q^{x+1} - q^{x-1}$ affine points. As $|S \setminus Y| = q^{x+1} + q^x$, then $S \setminus Y$ contains less than $q^x + q^{x-1}$ points that are not affine. There are q^{x+1} points in S_P that do not lie in $U_P = \langle P, Y \rangle$. Thus there exists an affine point P' in $S \setminus U_P$. Then $S = S_P = \langle Y, P, P' \rangle$. As $P' \in S_P$, then $P \in S_{P'}$ (Lemma 4.5) and hence $S_{P'} = \langle Y, P, P' \rangle = S$. If R is any affine point of S, then $R \in S_P, S_{P'}$ and hence $P, P' \in S_R$, so that $S_R = \langle Y, P, P' \rangle = S$.

Let Q be a point of Y. As Q lies on $q^x + q^{x-1}$ lines that contain a point of $S \setminus Y$, then S contains a line l with $l \cap Y = Q$ and such that all points of $l \setminus \{Q\}$ are affine. Then $S = S_R$ for all points $R \in l \setminus \{Q\}$. Hence, every B-line that contains a point R of $l \setminus \{Q\}$ is contained in $S_R = S$. If we apply Lemma 4.4 (b) to one of the points $R \in l \setminus \{Q\}$, we see that there exists at least

$$c := q^x \Theta_{s-1} - \frac{q^{x-1}}{q-2}$$

planes π with $\pi \cap S = l$ such that π contains a B-line. This gives at least c lines of B that meet l and do not lie in S. Hence these lines pass through Q. This proves (a). Part (a) then implies that Y meets at least $\Theta_{x-1}c$ lines of B. Using the upper bound (2) for $|B|$, the assertion of (b) follows. ∎

Lemma 4.8. *If P is a point not on Y, then P lies on at least q^x lines of B that miss Y. If equality holds, then there exists a subspace S_P of dimension $x+1$ such that $\langle P,Y\rangle \subseteq S_P$ and such that all B-lines on P are contained in S_P.*

Proof. Each of the q^x point of $\langle P,Y\rangle \setminus Y$ lies on at least q^x lines of B and thus on at least $q^x - \Theta_{x-1}$ lines of B that miss Y. Thus, there exist at least $q^x(q^x - \Theta_{x-1})$ lines in B that meet $\langle P,Y\rangle$ and that miss Y. Then Lemma 4.7 (b), shows that at most

$$n := \frac{q^{2s}-1}{q^2-1}q^{2x}$$

lines of B are skew to $\langle P,Y\rangle$.

In the rest of this proof we shall consider subspaces U of dimension $2s-1$ satisfying $U \cap \langle P,Y\rangle = P$, and lines $h \in B$ satisfying $h \cap \langle P,Y\rangle = \emptyset$. There are $q^{x(2s-1)}$ such subspaces U, and at most n such lines h. We count in two ways the number of incident pairs (h,U) for these lines h and subspaces U. Every line h occurs in $q^{x(2s-3)}$ such pairs. So the number of pairs (h,U) is at most $nq^{x(2s-3)}$.

The subspaces U we consider satisfy $U \cap \langle P,Y\rangle = P$. Hence a B-line in such a subspace U contains P or is disjoint to $\langle P,Y\rangle$, and in the latter case it is one of the lines h. Result 3.2 (a) shows that each subspace U contains at least $(q^{2s}-q^2)/(q^2-1)$ lines h. Let f be the number of subspaces U such that P lies on no B-line on U. Result 3.2 (b) shows that these f subspaces U contain at least $(q^{2s}-q^2)/(q^2-1)+\Theta_{s-1}$ lines h. Thus the number of pairs (h,U) is at least

$$q^{x(2s-1)}\frac{q^{2s}-q^2}{q^2-1} + f\Theta_{s-1}.$$

Since this number is at most $nq^{x(2s-3)}$, it follows that $f\Theta_{s-1} \le q^{x(2s-1)}$.

In order to prove the lemma, we may assume that P lies on at most q^x lines of B that miss Y. Let K be the set consisting of the B-lines on P that miss Y and of the Θ_{x-1} lines on P that meet Y. Then $|K| \le \Theta_x$. We have to show that K consists of the lines on P in a subspace of dimension $x+1$. Assume that this is not true. Apply Lemma 2.7 to the quotient geometry at P, to see that there exist at least $q^{x(2s-3)}(q^x-1)$ subspaces V of dimension $2s-1$ on P that contain no line of K. The hypotheses of Lemma 2.7 are fulfilled as the Θ_{x-1} lines of the subspace $\langle P,Y\rangle$ on P lie in K. Such a subspace V satisfies $V\cap Y=\emptyset$ and thus $V\cap\langle P,Y\rangle=P$. Hence these subspaces V are subspaces U considered above that have the additional property that P does not lie on a B-line of U. Therefore $f\geq q^{x(2s-3)}(q^x-1)$. If we compare this to the above upper bound for f using $s\geq x+2$, we obtain a contradiction. ∎

Definition. Points that are not in Y and that lie on exactly q^x lines of B that miss Y will be called *quasiaffine* points. If P is a quasiaffine point, then we shall denote the subspace of dimension $x+1$ containing all the B-lines on P by S_P.

Lemma 4.9. *Let P be a quasiaffine point.*

(a) The number of B-lines that meet $S_P\setminus Y$ in a unique point is less than $q^{2x}/2$.

(b) If h is a line with $h\notin B$ and $P\in h\subseteq S_P$, then at there are at least $q^x\Theta_{s-1}-q^{x-1}/(q-2)$ planes π with $\pi\cap S_P=h$ and such that π contains a B-line.

Proof. Let c_0 be the number of B-lines disjoint to S_P, let c_2 be the number of B-lines contained in $S_P\setminus Y$, and let c_1 be the number of B-lines that meet $S_P\setminus Y$ in a unique point. Lemma 4.7 (b) shows that

$$c_0+c_1+c_2\leq q^{2x}\frac{q^{2s}-1}{q^2-1}+\frac{q^{2x-1}}{q-2}\,.$$

As each of the $(q+1)q^x$ points of $S_P\setminus Y$ lies on at least q^x lines of B that miss Y, we have $(q+1)c_2+c_1\geq(q+1)q^{2x}$. Using this as a lower bound for c_2 in the above inequality gives

$$c_0+\frac{q}{q+1}c_1\leq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+\frac{q^{2x-1}}{q-2}\,. \tag{17}$$

(a) There exist exactly $q^{(x+1)(2s-2)}$ subspaces U of dimension $2s-2$ with $U\cap S_P=P$. Such a subspace U does not contain B-lines on P. Apply Result 3.2 to the quotient geometry at P, to see that U contains at least $\frac{q^{2s-2}-1}{q^2-1}$ lines of B. As every B-line that is disjoint to S_P occurs in $q^{(x+1)(2s-4)}$ such subspaces U, it follows that

$$c_0\geq\frac{q^{(x+1)(2s-2)}\frac{q^{2s-2}-1}{q^2-1}}{q^{(x+1)(2s-4)}}=q^{2x}\frac{q^{2s}-q^2}{q^2-1}\,.$$

This and (17) imply $c_1\leq q^{2x-2}(q+1)/(q-2)$. This proves (a).

(b) Let r be the number of planes π with $\pi\cap S_P=h$ that contain a B-line. For the proof of (b) we may assume that $r=q^x\Theta_{s-1}-c$ with $c\geq 0$. Lemma 3.3 applied to the quotient geometry at P shows

$$c_0\geq q^{2x}\frac{q^{2s}-q^2}{q^2-1}+cq^x.$$

As $c_1\geq 0$, this and (17) imply $c\leq q^{x-1}/(q-2)$. ∎

Lemma 4.10. *If P and R are quasiaffine points with $P,R\notin Y$, then $S_P=S_R$ or $S_P\cap S_R=Y$.*

Proof. Put $U:=S_P\cap S_R$. As $Y\subseteq U$, we have to show that $\dim(U)\neq x$. Assume on the contrary that $\dim(U)=x$. Each of the q^x points of $U\setminus Y$ lies on at least q^x lines of B that miss Y. This gives rise to q^{2x} lines that meet U but not Y. These lines meet U in a unique point. Hence, none of them lies in S_P and S_R, so we may assume that $q^{2x}/2$ of them meet $S_P\setminus Y$ but are not contained in S_P. This contradicts Lemma 4.9. ∎

Lemma 4.11. *Every point outside Y is quasiaffine. If P is a point of Y, then there exists a subspace F_P of dimension $s+x$ with $Y\subseteq F_P$ and such that the B-lines on P are the lines of F_P on P that do not lie in Y.*

Proof. Part I: For every point $A\notin Y$, let q^x+d_A be the number of B-lines on A that miss Y. Then Let c_0 be the number of B-lines that miss Y. Then

$$c_0=\frac{1}{q+1}\sum_{A\notin Y}(q^x+d_A)=\frac{q^{2s}-1}{q^2-1}q^{2x}+\frac{1}{q+1}\sum_{P\notin Y}d_A. \tag{18}$$

Lemma 4.7 gives

$$\sum_{A\notin Y}d_A\leq\frac{(q+1)q^{2x-1}}{q-2}<2q^{2x-1}. \tag{19}$$

Notice that $d_A\geq 0$ with equality iff A is quasiaffine. For points $X\in Y$, let r_X be the number of B-lines h satisfying $h\cap Y=X$. Put $r:=\min\{r_X\mid X\in Y\}$. Then Y meets at least $\Theta_{x-1}r$ lines of B. Hence $r\Theta_{x-1}+c_0\leq|B|$. Using (2) and (18), it follows

$$r\leq q^{x-1}\Theta_s \tag{20}$$

with equality if and only if (2) is satisfied with equality, every point outside Y is quasiaffine, no line of B is contained in Y, and $r_X=r$ for all $X\in Y$.

In Part II of the proof we show for every point $P\in Y$ with $r=r_P$ the existence of a subspace F_P of dimension $s+x$ on P such that all lines of F_P

that pass through P but do not lie in Y are B-lines. This gives $r \geq q^{x-1}\Theta_s$ with equality only if $Y \subseteq F$. Then it follows that $r = q^{x-1}\Theta_s$ and $Y \subseteq F_P$. This proves the lemma, since now $r_X = r$ for all $X \in Y$.

Part II: Let $P \in Y$ with $r = r_P$. Consider the $q^{2s(x-1)}$ subspaces U of dimension $2s$ satisfying $U \cap Y = P$. By Result 3.1, each such subspace U contains at least $q^2\frac{q^{2s}-1}{q^2-1} + \Theta_s$ lines of B. Suppose that e of these subspaces contain more than this many lines of B. We count pairs (U,h) where U is a subspace of dimension $2s$ with $U \cap Y = P$ and h is a B-line of U. Every B-line h with $h \cap Y = P$ occurs in $q^{(x-1)(2s-1)}$ such pairs and every B-line h with $h \cap Y = \emptyset$ occurs in $q^{(x-1)(2s-2)}$ such pairs. Hence

$$q^{2s(x-1)}\left(q^2\frac{q^{2s}-1}{q^2-1} + \Theta_s\right) + e \leq rq^{(x-1)(2s-1)} + c_0 q^{(x-1)(2s-2)}.$$

Combining this with $c_0 \leq |B| - \Theta_{x-1}r$ and the upper bound (2) for $|B|$, we obtain

$$q^{2s(x-1)}\Theta_s + e \leq rq^{(x-1)(2s-1)} + q^{(x-1)(2s-2)}\left(\Theta_{x-1}q^{x-1}\Theta_s - \Theta_{x-1}r\right).$$

This gives

$$e \leq q^{(x-1)(2s-2)}\Theta_{x-2}\left(q^{x-1}\Theta_s - r\right).$$

As $r \geq q^x\Theta_{s-1} - q^{x-1}/(q-2)$ (Lemma 4.7 (a)), it follows that $e \leq 2q^{2s(x-1)-1}$.

Denote by $\mathcal{U}$ the set consisting of the subspaces U of dimension $2s$ that meet Y in P and that contain precisely $q^2\frac{q^{2s}-1}{q^2-1} + \Theta_s$ lines of B. We have just shown that $|\mathcal{U}| \geq q^{2s(x-1)}(1-2/q)$. The rest of the proof is divided into several parts.

(a) If $U \in \mathcal{U}$, then U contains a point R_U and a subspace V_U of dimension $s+1$ with $R_U \in V_U$ such that the B-lines of U on R_U are the Θ_s lines of V_U on R_U. Every point of U other than R_U lies on q or $q+1$ lines of U that are in B. This follows from Result 3.1 applied to U.

(b) We have $R_U = P$ for all $U \in \mathcal{U}$.

To see this, notice first that a point R with $R \notin Y$ occurs in $q^{(2s-1)(x-1)}$ subspaces U of dimension $2s$ with $U \cap Y = P$. Assume that $R_U \neq P$ for all subspaces $U \in \mathcal{U}$. Then we obtain at least

$$\frac{|\mathcal{U}|}{q^{(2s-1)(x-1)}} \geq (q-2)q^{x-2}$$

different points R_U. All these points are not in Y and lie on at least Θ_s lines of B. Then (19) gives

$$(q-2)q^{x-2}(\Theta_s - q^x) < 2q^{2x-1}.$$

This contradicts (1). Hence, there exists a subspace $U\in\mathcal{U}$ with $R_U=P$. For $U'\in\mathcal{U}$ the subspace $U'\cap V_U$ has dimension

$$\dim(U')+\dim(V_U)-\dim(\langle U',V_U\rangle)\geq 2s+s+1-(2s-1+x)=s+2-x.$$

Also $R_U=P\in U'\cap V_U$. Therefore, at least Θ_{s+1-x} of the lines of U on $R_U=P$ lie in U'. As these lines lie in B, it follows that P lies in U' on at least $\Theta_{s+1-x}>q+1$ lines of B. Then (a) implies that $P=R_{U'}$.

(c) Given a set of at most Θ_{s+x-2} lines of B on P, there exists a B-line on P that is not in this set and that occurs in at least $q^{(2s-1)(x-1)}(1-3/q)$ of the subspaces U of $\mathcal{U}$.

To see this, we count incident pairs (h,U) with B-lines h on P and subspaces $U\in\mathcal{U}$. Each $U\in\mathcal{U}$ occurs in Θ_s of these pairs. Each line of B on P occurs either in no such pair (when it is contained in Y), or in at most $q^{(2s-1)(x-1)}$ (since this is the number of subspaces U of dimension $2s$ satisfying $U\cap Y=P$ and containing a given line of h with $h\cap Y=P$). Thus, if z is the largest number of subspaces $U\in\mathcal{U}$ containing a B-line on P that is not in the given set, then

$$\Theta_{s+x-2}q^{(2s-1)(x-1)}+(r-\Theta_{s+x-2})z\geq|\mathcal{U}|\Theta_s$$

where r is the number of B-lines on P. As $r\leq q^{x-1}\Theta_s$ and $|\mathcal{U}|\geq q^{2s(x-1)}(1-2/q)$, it follows that $z\geq q^{(2s-1)(x-1)}(1-3/q)$.

(d) For $i:=0,\dots,x-1$, there exists a subspace F_i of dimension $s+1+i$ on P such that at least $\Theta_{s+i}-(4i)q^{s+i-1}$ of the lines of F_i on P are in B.

This we prove using induction on i. For $i=0$, we take a $U\in\mathcal{U}$ and put $F_0:=V_U$. Then F_0 has dimension $s+1$ and all lines of F_0 on P are in B. Suppose now that $0\leq i\leq x-2$ and that we have already found a subspace F_i with the required properties. We shall construct F_{i+1}.

By (c), we find a B-line h on P that is not in F_i such that the set $\mathcal{U}_h:=\{U\in\mathcal{U}\mid h\subseteq U\}$ contains at least $q^{(2s-1)(x-1)}(1-3/q)$ elements. Define $F_{i+1}:=\langle F_i,h\rangle$. Then F_{i+1} has dimension $s+2+i$. For $U\in\mathcal{U}_h$ we have

$$\dim(F_i\cap U)\geq\dim(F_i)+\dim(U)-(2s-1+x)=s+2+i-x.$$

We want to show that for most subspaces $U\in\mathcal{U}_h$, the subspace $F_i\cap U$ is contained in V_U. First recall from (a) that all B-lines of $F_i\cap U$ on P lie in V_U. Hence, if $F_i\cap U\not\subseteq V_U$, then at least $q^{s+1+i-x}$ of the lines of $F_i\cap U$ on P do not lie in B. Suppose that this happens for z of the subspaces $U\in\mathcal{U}_h$. Then the number of pairs (l,U) with subspaces $U\in\mathcal{U}_h$ and lines l satisfying $l\notin B$ and $P\in l\subseteq U\cap F_i$ is at least $zq^{s+1+i-x}$. On the other hand, the induction hypothesis says that at most $4iq^{s+i-1}$ lines of F_i on P do not lie B. Also a line l of F_i on P occurs in at most $q^{(2s-2)(x-1)}$ subspaces of $\mathcal{U}_h$ (this is the number of subspaces of dimension $2s$ that meet Y in P and contain a given plane π satisfying $\pi\cap Y=P$). Thus comparing the two bounds for the number of pairs gives

$$zq^{s+1+i-x} \le 4iq^{s+i-1}q^{(2s-2)(x-1)},$$

that is $z\le 4iq^{(2s-1)(x-1)-1}$. As $|\mathcal{U}_h|\ge q^{(2s-1)(x-1)}(1-3/q)$, it follows that for at least $d:=q^{(2s-1)(x-1)}(1-(4i+3)/q)$ subspaces $U\in\mathcal{U}_h$, we have $F_i\cap U\subseteq V_U$ and thus $\dim(F_i\cap V_U)\ge s+2+i-x$.

Consider one of these d subspaces $U\in\mathcal{U}_h$. As $h\in B$ and $h\subseteq U$, we have $h\subseteq V_U$. Thus,

$$\dim(V_U\cap F_{i+1}) = \dim(V_U\cap F_i)+1 \ge s+3+i-x.$$

This implies that at least $q^{s+2+i-x}$ of the B-lines of U on P lie in F_{i+1} but not in F_i; one of these lines is h. Since every line other than h on P that lies in F_{i+1} but not in F_i occurs in at most $q^{(2s-2)(x-1)}$ subspaces $U\in\mathcal{U}_h$, it follows that the number t of B-lines on P that lie in F_{i+1} but not in F_i satisfies

$$(t-1)q^{(2s-2)(x-1)} \ge d(q^{s+2+i-x}-1).$$

Thus

$$t > q^{x-1}(q^{s+2+i-x}-1)(1-\frac{4i+3}{q}).$$

As F_i contains $\Theta_{s+i}-4iq^{s+i-1}$ lines of B on P, it follows that the number of B-lines of F_{i+1} on P is at least

$$q^{x-1}(q^{s+2+i-x}-1)(1-\frac{4i+3}{q})+\Theta_{s+i}-4iq^{s+i-1}.$$

This number is larger than $\Theta_{s+i+1}-4(i+1)q^{s+i}$.

(e) There exists a subspace F of dimension $s+x$ on P containing at least $\Theta_{s+x-1}-3q^{2x-1}$ lines of B on P.
By (d) some subspace F of dimension $s+x$ contains at least $\Theta_{s+x-1}-4(x-1)q^{s+x-2}$ lines of B on P. Assume that more than $3q^{2x-1}$ lines of F on P are not in B. At least $2q^{2x-1}$ of these do not lie in Y. Then (19) implies that one of these lines, call it h, will have the property that all points of $h\setminus\{P\}$ are quasiaffine. This implies that there exists a subspace S of dimension $x+1$ on $\langle h,Y\rangle$ such that $S_R=S$ for all $R\in h\setminus\{P\}$. Then every B-line that contains a point of $h\setminus\{P\}$ lies in S.

As P lies on Θ_{x-2} lines of Y, then (20) shows that P lies on at most Θ_{s+x-1} lines of B. We know that at most $4(x-1)q^{s+x-2}$ of the B-lines on

P do not lie in F. As $h\subseteq F$, it follows that h lies in at most $\Theta_{s+x-2}+4(x-1)q^{s+x-2}$ planes that contain one of the B-lines on P. Thus, there exist at most this many planes π on h that contain a B-line and satisfy $\pi\cap S=h$. This contradicts Lemma 4.9 (b).

(f) All lines of F on P that are not contained in Y lie in B.
Assume on the contrary that F contains a line h with $h\cap Y=P$ and $h\notin B$. Consider the set $\mathcal{S}$ consisting of the s-subspaces S with $S\cap F=h$. Then $|\mathcal{S}|=q^{(s+x-1)(s-1)}$. Also, each line l with $|l\cap F|=|l\cap h|=1$ lies in $q^{(s+x-1)(s-2)}$ elements of $\mathcal{S}$, and a line that is skew to F lies in $q^{(s+x-1)(s-3)}$ elements of $\mathcal{S}$. Every $S\in\mathcal{S}$ has dimension s and contains therefore a B-line l_S. We consider different possibilities for the location of S and l_S.

Case 1. We have $S\cap Y\neq P$. Then S contains one of the Θ_{x-2} lines g of Y on P. Such a line g lies either in zero or $q^{(s+x-1)(s-2)}$ elements of $\mathcal{S}$ (zero in the case that g is contained in F). This shows that Case 1 can happen for at most $\Theta_{x-2}q^{(s+x-1)(s-2)}\leq q^{(s+x-1)(s-1)-2}$ elements S of $\mathcal{S}$.

Case 2. $S\cap Y=P$ and the line l_S meets F. Then $|l_S\cap F|=|l_S\cap h|=1$. We find an upper bound for the number of B-lines that meet h but are not contained in F. As P lies on at most Θ_{s+x-1} lines of B, Step (e) shows that P lies on at most $3q^{2x-1}$ such lines. If we denote the number of B-lines on a point $Z\in h\setminus\{P\}$ by q^x+d_Z, then $q^{x+1}+\sum_{P\neq Z\in h}d_Z$ lines of B meet $h\setminus\{P\}$; by (19) shows that this number is at most $q^{x+1}+2q^{2x-1}$. Thus, at most $q^{x+1}+5q^{2x-1}$ of the B-lines that meet h do not lie in F. As each of them lies in $q^{(s+x-1)(s-2)}$ elements of $\mathcal{S}$, we see that Case 2 can occur for at most

$$(q^{x+1}+5q^{2x-1})q^{(s+x-1)(s-2)}\leq 6q^{2x-1}q^{(s+x-1)(s-2)}\leq 6q^{(s+x-1)(s-1)-2}$$

elements S of $\mathcal{S}$.

Case 3. We have $S\cap Y=P$, $l_S\cap F=\emptyset$ and l_S contains a quasiaffine point. As $S\cap Y=P$ and $P\notin l_S$, then $l_S\cap Y=\emptyset$. The plane $\pi_0:=\langle P,l_S\rangle$ satisfies $\pi_0\cap F=P$. If A_0 is a quasiaffine point of l_S, then $l_S\in B$ implies that $l_S\subseteq S_{A_0}$ so that $S_{A_0}=\langle Y,l_S\rangle$. Thus $S\cap S_{A_0}$ contain the plane π_0 that satisfies $\pi_0\cap F=P$.

Recall that each quasiaffine point A gives rise to a subspace S_A of dimension $x+1$ and that different subspaces S_A meet in Y. The number of different subspaces S_A is thus at most $(q^{2s}-1)/(q^2-1)$. Each such subspace S_A contains $\Theta_x\Theta_{x-1}/(q+1)$ planes π on P. If such a plane π satisfies $\pi\cap F=P$, then π lies in $q^{(s+x-1)(s-3)}$ elements S of $\mathcal{S}$. Putting this information together, we see that the number of subspaces $S\in\mathcal{S}$ that satisfy the hypotheses of Case 3 is at most

$$\frac{q^{2s}-1}{q^2-1}\cdot\frac{\Theta_x\Theta_{x-1}}{q+1}\cdot q^{(s+x-1)(s-3)}\leq 2q^{(s+x-1)(s-1)-2}.$$

Case 4. We have $S\cap Y=P$, $l_S\cap F=\emptyset$ and l_S contains no quasiaffine point. Thus S contains $q+1$ points R that are not quasiaffine and satisfy $R\notin F,Y$. By (19), there exist less than $2q^{2x-1}$ points R outside Y that are not quasiaffine. Also every point $R\notin F$ occurs in exactly $q^{(s+x-1)(s-2)}$ elements of $\mathcal{S}$. It follows that Case 4 occurs for less than $2q^{2x-1}q^{(s+x-1)(s-2)}/(q+1)<q^{(s+x-1)(s-1)-2}$ elements of S of $\mathcal{S}$.

Combining Cases 1, 2, 3, 4, we find that $|\mathcal{S}|\le 11q^{(s+x-1)(s-1)-2}$. But $|\mathcal{S}|=q^{(s+x-1)(s-1)}$, a contradiction. This completes the proof of (f) and of the lemma. ∎

Every point A not in Y is quasiaffine, and gives therefore rise to a subspace S_A of dimension $x+1$ containing Y, such that the B-lines on A that miss Y are precisely the q^x lines of S_A on A that miss Y. Also for two points A and A' outside Y we have either $S_A=S_{A'}$ or $S_A\cap S_{A'}=Y$ (Lemma 4.10). Thus, if $\mathcal{S}$ is the set consisting of the different subspaces S_A, then $\mathcal{S}$ is a line spread in the quotient geometry on Y, which is a PG$(2s-1,q)$.

Lemma 4.12. *The line-spread $\mathcal{S}$ is geometric.*

Proof. As mentioned in the introduction, it suffices to show that every s-subspace of the PG$(2s-1,q)$ (which is the quotient geometry on Y) contains a line of $\mathcal{S}$. Back in the original space PG$(2s-1+x,q)$, this means that every subspace T of dimension $s+x$ on Y contains a subspace S_A for some point $A\notin Y$. To see this, let S be a subspace of dimension s of T with $S\cap Y=\emptyset$. Then S contains a line l of B. Then $l\subseteq S_A$ for all $A\in l$, which implies that $S_A=\langle l,Y\rangle\subseteq T$. ∎

It remains to show that the sets F_P with $P\in Y$ behave as described in Example 1.1. For this we need a lemma and a result. The result follows from Theorem 1 of [8].

Result 4.13. *Suppose T is a set of points of* PG(d,q) *that meets every subspace of dimension n, where $n<d$ and $q\ge 5$. If $|T|<\Theta_{d-n}+\sqrt{q}q^{d-n-1}$, then T contains a subspace of dimension $d-n$.*

Lemma 4.14. *Consider in* PG$(2s-1,q)$ *a set $\mathcal{S}$ of mutually skew lines and also $q+1$ subspaces $F_0,\dots,F_q$ of dimension s. Put $B':=F_0\cup\ldots\cup F_q$. Suppose that every subspace of dimension $s-2$ contains a point of B' or a line of $\mathcal{S}$. Then there exists a subspace E of dimension $s-1$ and a subspace G of dimension $s+1$ such that the F_i are the $q+1$ subspaces that lie between E and G.*

Proof. We may assume that $\mathcal{S}$ is minimal with respect to the property that every subspace of dimension $s-2$ contains a point of B' or a line of $\mathcal{S}$. We show first that this implies that $\mathcal{S}=\emptyset$.

Assume that this is not the case and consider a line $l\in\mathcal{S}$. The assumption just made on $\mathcal{S}$ implies that there exists a subspace U of dimension $s-2$ on l that contains no other element of $B'\cup\mathcal{S}$. As U has dimension $s-2$, it is contained in Θ_s subspaces of dimension $s-1$; also U meets at most Θ_{s-2} elements of $\mathcal{S}$ (since the lines of $\mathcal{S}$ are skew). It follows that there exists a subspace V of dimension $s-1$ on U such that l is the only line of $\mathcal{S}$ contained in V. As $U\cap F_i=\emptyset$ for all i, then $V\cap F_i$ is a point P_i. As every subspace of dimension $s-2$ contains an element of $B'\cup\mathcal{S}$, then every subspace of dimension $s-2$ of V contains l or one of the points $P_0,\dots,P_q$. This implies that for each point $P\in l$, every subspace of dimension $s-2$ of V contains a point of $\{P,P_0,\dots,P_q\}$. The previous result implies then that $\{P,P_0,\dots,P_q\}$ contains a line. This holds for any choice of the point P of l. As l lies in U and the points P_i lie in $V\setminus U$, it follows that $h:=\{P_0,\dots,P_q\}$ must be a line. Then this line h of V must meet the hyperplane U of V. This is a contradiction, since no point P_i lies in U.

Thus every $(s-2)$-subspace contains an element of B'. As $|B'|\le(q+1)\Theta_s$, the previous result implies that B' contains a subspace G of dimension $s+1$. Then G is the union of its $q+1$ subspaces $F_i\cap G$. The previous result (in its dual form) implies then that the F_i are contained in G and form a dual line, that is they all contain a subspace E of dimension $s-1$. ∎

Lemma 4.15. *(a) If l is a line of Y, then there exists a subspace E_l of dimension $s+x-1$ and a subspace G_l of dimension $s+x+1$ such that the subspaces F_P with $P\in l$ are the $q+1$ subspaces of dimension $s+x$ that lie between E_l and G_l. This implies that for distinct points $P\in Y$ also the subspaces F_P are distinct.*

(b) If G is the subspace generated by all F_P with $P\in Y$, then $\dim(G)\ge s+2x-1$.

Proof. (a) Recall that every point $P\in Y$ gives rise to a subspace F_P of dimension $s+x$ with $Y\subseteq F_P$ such that the B-lines on P are the lines of F_P on P that do not lie in Y. Also, for every B-line h that is skew of Y we have $S_A=\langle Y,h\rangle$ for all $A\in h$ (since the points A of h are quasiaffine), so that $\langle Y,h\rangle$ is an element of $\mathcal{S}$.

Consider a complement T of Y, that is $\dim(T)=2s-1$ and $T\cap Y=\emptyset$. If $P\in l$, then $Y\subseteq F_P$ and $F_P\cap T$ has dimension s. If $S\in\mathcal{S}$, then $S\cap T$ is a line. The lines $S\cap T$ for $S\in\mathcal{S}$ form a line-spread of T, which we denote by $\mathcal{S}_T$.

Consider a subspace S' of dimension $s-2$ of T. Then $S:=\langle S',l\rangle$ has dimension s. Thus S contains a B-line h. Either h contains some point $P\in l$ and lies then in F_P, or h is skew to Y and then $\langle Y,h\rangle\in\mathcal{S}$. In the first case S' contains a point of $F_P\cap T$ for some $P\in l$. In the second case S' contains the line $S\cap T$ for some $S\in\mathcal{S}$.

It follows therefore from Lemma 4.14 that there exist a subspaces E,G of T with $\dim(E)=s-1$ and $\dim(G)=s+1$ such that the subspaces $F_P\cap T$ for $P\in l$ lie between E and G. Then $E_l:=\langle E,Y\rangle$ and $G_l:=\langle G,Y\rangle$ are the subspaces we are looking for.

(b) Assume on the contrary that there exists a subspace M of dimension $s+2x-2$ containing all subspaces F_P. We shall derive a contradiction. We use the notation of the proof of part (a). Then $M':=M\cap T$ has dimension $s+x-2$. A B-line h on a point P of Y lies in F_P (but not in Y) and thus satisfies $h\cap M'\neq\emptyset$. In other words, every B-lines that misses M' is skew to Y.

Consider a complement U of M' in T. Then U has dimension $s-x$ and $\bar{U}:=\langle U,Y\rangle$ has dimension s. Also $\bar{U}\cap M'=\emptyset$. As $\bar{U}$ has dimension s, then $\bar{U}$ contains a B-line h. As $h\cap M'=\bar{U}\cap M'=\emptyset$, then h is skew to Y. Hence $S:=\langle Y,h\rangle$ is an element of $\mathcal{S}$. Therefore U contains the line $S\cap T$ of $\mathcal{S}_T$.

Hence, every complement U of M' in T contains a line of $\mathcal{S}_T$. By Lemma 2.1, there are $q^{(s+x-1)(s-x+1)}$ such complements U. If $h\in\mathcal{S}_T$, then h lies in no such complement if $h\cap M'\neq\emptyset$, and h lies by Lemma 2.1 in $q^{(s+x-1)(s-x-1)}$ such complements if $h\cap M'=\emptyset$. It follows that

$$|\mathcal{S}_T|q^{(s+x-1)(s-x-1)}\geq q^{(s+x-1)(s-x+1)}.$$

Hence $|\mathcal{S}_T|\geq q^{2(s+x-1)}$. As $|\mathcal{S}_T|=\Theta_{2s-1}/(q+1)$, this is a contradiction. ∎

Lemma 4.16. *There exist subspaces E and G with $\dim(E)=s+x-1$ and $\dim(G)=s+2x-1$ such that $E\subseteq F_P\subseteq G$ for all $P\in Y$. Moreover, the map $P\to F_P/E$ is an isomorphism of Y onto G/E.*

Proof. For $x=2$, this has been proved in part (a) of the previous lemma. We assume $x\geq 3$ from now on. By the previous Lemma, we see that $F_P\cap F_{P'}$ has dimension $s+x-1$ for any two distinct points $P,P'\in Y$. This implies that there either exists a subspace E of dimension $s+x-1$ contained in all F_P, or that there exists a subspace M of dimension $s+x+1$ containing all F_P. As $x\geq 3$, part (b) of the previous lemma shows that the second case is not possible. Hence there exists a subspace E of dimension $s+x-1$ with $E\subseteq F_P$ for all $P\in Y$.

Let G be the subspace generated by all F_P with $P\in Y$. Then $\dim(G)\geq s+2x-1$ by the previous lemma. For each F_P we can view F_P/E as a point in the quotient geometry at G/E. Consider the map $P\to F_P/E$ from Y to G/E. The previous Lemma implies that this map is injective and that collinear points of Y are mapped onto collinear points of G/E. The image of the map is thus a subspace of G/E. As G is generated by the F_P with $P\in Y$, it follows that this map is an isomorphism. Hence $\dim(G/E)=\dim(Y)=x-1$, that is $\dim(G)=\dim(E)+x=s+2x-1$. ∎

The preceding lemma completes the proof that B has the structure described in Example 1.1.

References

[1] A. Beutelspacher: On t-covers in finite projective spaces, *Journal of Geometry* **12** (1979), 10–16.

[2] A. Blokhuis: Blocking sets in Desarguesian planes; in *Combinatorics, Paul Erdős is Eighty*, volume 2, pages 133–155, János Bolyai Math. Soc., Budapest, 1996 (Keszthely, 1993).

[3] A. Beutelspacher and J. Ueberberg: A characteristic property of geometric t-spreads in finite projective spaces, *Europ. J. Comb.* **12** (1991), 277–281.

[4] R. C. Bose and R. C. Burton: A characterization of Flat Spaces in a Finite Geometry and the uniqueness of the Hamming and the MacDonald Codes, *J. Comb. Th.* **1** (1966), 96–104.

[5] J. Eisfeld: On smallest covers of finite projective spaces, *Archiv der Mathematik* **68**, (1997), 77–80.

[6] J. Eisfeld and K. Metsch: Blocking s-dimensional subspaces by lines in PG$(2s, q)$, *Combinatorica* **17** (1997), 151–162.

[7] J. W. P. Hirschfeld: *Projective Geometries over Finite Fields*, 2$^{\text{nd}}$ edition, Claredon Press, Oxford, 1998.

[8] M. Huber: Baer cones in finite projective spaces, *J. Geom.* **28** (1987), 128–144.

[9] K. Metsch: Bose-Burton Type Theorems for Finite Projective, Affine and Polar Spaces; in *Surveys in Combinatorics*, volume 267 of *London Math. Soc. Lecture Notes Series*, pages 137–166, Cambridge University Press, Cambridge, 1999.

[10] T. Szőnyi: Blocking sets in Desarguesian affine and projective planes, *Finite Fields Appl.* **3** (1997), 187–202.

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